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1978 J. Phys. A: Math. Gen. 11 27

(http://iopscience.iop.org/0305-4470/11/1/007)

J. Phys. A: Math. Gen., Vol. 11, No. 1, 1978. Printed in Great Britain. © 1978

Disruption of wavefronts: statistics of dislocations in incoherent Gaussian random waves

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Received 1 August 1977

Abstract. Wavefront dislocations—i.e. singularities of the phase of a wave ψ in the form of moving lines in space where $|\psi|$ vanishes—are studied for initially plane waves that have passed through a random space and time-dependent phase-changing screen. For transmitted waves that are Gaussian random, incoherent, quasi-monochromatic and paraxial the following quantities are calculated in terms of the statistics of the phase screen: *dislocation densities*, i.e. the average number of dislocation lines piercing unit area of variously-oriented surfaces, and *dislocation fluxes*, i.e. the average number of dislocation lines crossing unit length of variously-directed lines in unit time. For a 'corrugated' screen (i.e. one where the phase varies only in one direction) all dislocations are of 'edge' type. As the statistics of the screen are made more isotropic the dislocations retain predominantly edge character if the screen is moving fast enough, but become predominantly of 'screw' character if the screen is static.

1. Introduction

A wave ψ travelling through space often contains moving 'dislocation lines' (Nye and Berry 1974) where the wavefronts (surfaces of constant phase) have singularities. On dislocations $|\psi|$ vanishes and near dislocations the vector field formed by the gradient of the phase of ψ is that of a vortex (see also Dirac 1931, Riess 1970a, b, 1976, Hirschfelder *et al* 1974a, b, Hirschfelder and Tang 1976a, b). My purpose here is to study some statistical properties of the tangle of dislocation lines in waves that are random—for example as a result of traversing an irregular refracting medium or being reflected from a rough surface. Walford *et al* (1977) have made a direct observation of dislocations in random radio waves reflected from the subglacial topography of Devon Island in the Arctic.

The degree of disruption of wavefronts is indicated by the densities and fluxes of dislocations near any event (x, y, z, t) in the wave. Let i and j represent x, y or z. Then the dislocation density N_{ij} is defined as the average number of dislocation lines piercing unit area of the ij plane. The dislocation flux N_{it} is defined as the average number of dislocation lines crossing unit length of the i direction in unit time. Each dislocation is counted once whatever its sign or strength (Nye and Berry 1974) so that N_{ij} and N_{it} are never negative. It will be convenient to denote densities and fluxes by $N_{\alpha\beta}$ where α and β represent x, y, z or t (and $\alpha \neq \beta$).

For a general random wave ψ the calculation of $N_{\alpha\beta}$ would present formidable difficulties. Therefore I employ instead what seems to be the simplest non-trivial model. A monochromatic plane wave of unit amplitude and frequency ω travels in the z direction with speed c and encounters a transparent space-and-time-dependent screen that imposes on it a phase $\phi(\boldsymbol{R}, t)$ in the plane $z=0$ ($\boldsymbol{R}$ denotes (x, y)). ϕ is a Gaussian random function of $\boldsymbol{R}$ and t whose gradients are small enough for all Fourier components of ψ to travel in directions making only small angles with the z axis and to have frequencies close to ω. Thus ψ is paraxial and quasi-monochromatic. Next, ψ itself will be assumed to be a Gaussian random function (Rice 1944, 1945, Longuet-Higgins 1956) of its variables. Finally, the variance $\bar{\phi}^2$ of the random phase screen will be assumed to exceed several squared radians, so that the mean value of ψ itself, namely $\exp(-\bar{\phi}^2/2)$ is negligible (in comparison with the RMS value of ψ) and ψ is incoherent. In the near field of such a screen the effects of focusing make the statistics of ψ strongly non-Gaussian (Berry 1977) but for sufficiently large z the assumed Gaussian statistics will always apply (Mercier 1962).

The outcome of the analysis is a set of simple formulae (equations (49)–(52)) for $N_{\alpha\beta}$ in terms of the statistical properties of the random phase screen, with rich physical content discussed in § 6.

2. Statistical formulae

Let the wave beyond the screen be written as

$$\psi(\boldsymbol{R}, z, t)=\xi(\boldsymbol{R}, z, t)+\mathrm{i}\eta(\boldsymbol{R}, z, t). \tag{1}$$

For events on a dislocation line, $|\psi|=0$ and so both the real and imaginary parts ξ and η must vanish. Now consider a small area $\mathscr{A}$ of the ij plane at $(\boldsymbol{R}, z, t)$. The number $N_{\mathscr{A}}$ of dislocation lines piercing $\mathscr{A}$ is

$$N_{\mathscr{A}}=\iint_{\mathscr{A}} \mathrm{d}i\,\mathrm{d}j\,\delta(\xi)\delta(\eta)|\xi_i\eta_j-\xi_j\eta_i|, \tag{2}$$

where the Jacobian factor between modulus signs ensures that each dislocation contributes $+1$ to $N_{\mathscr{A}}$, and the subscripts on ξ and η denote derivatives with respect to i or j. An analogous formula with i, t replacing i, j gives the number of dislocations crossing a length $\mathscr{L}$ of the i axis in time $\mathscr{T}$, if $\mathscr{L}\mathscr{T}$ replaces $\mathscr{A}$. Then the dislocation densities and fluxes $N_{\alpha\beta}$ are obtained by setting $\mathscr{A}$, $\mathscr{L}$ and $\mathscr{T}$ equal to unity and taking an ensemble average over the joint probability distribution P of ξ, η and their derivatives. This gives

$$N_{\alpha\beta}=\int\ldots\int \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\xi_\alpha\,\mathrm{d}\xi_\beta\,\mathrm{d}\eta_\alpha\,\mathrm{d}\eta_\beta\,\delta(\xi)\delta(\eta)|\xi_\alpha\eta_\beta-\xi_\beta\eta_\alpha|P(\xi, \eta, \xi_\alpha, \eta_\alpha, \xi_\beta, \eta_\beta). \tag{3}$$

(There can be higher-order zeros of ψ which are not dislocations (Nye and Berry 1974) but such non-generic events will not affect the statistics calculated here.)

By assumption, P is a joint Gaussian distribution in the six real random variables $\xi, \eta, \xi_\alpha, \eta_\alpha, \xi_\beta, \eta_\beta$. Denote by x the row vector

$$X\equiv(\eta, \xi_\alpha, \xi_\beta, \xi, \eta_\alpha, \eta_\beta), \tag{4}$$

and by Σ the matrix of correlations

$$\Sigma \equiv \begin{pmatrix} \overline{\eta^2} & \overline{\eta\xi_\alpha} & \overline{\eta\xi_\beta} & \overline{\eta\xi} & \overline{\eta\eta_\alpha} & \overline{\eta\eta_\beta} \\ \overline{\eta\xi_\alpha} & \overline{\xi_\alpha^2} & \overline{\xi_\alpha\xi_\beta} & \overline{\xi_\alpha\xi} & \overline{\xi_\alpha\eta_\alpha} & \overline{\xi_\alpha\eta_\beta} \\ \overline{\eta\xi_\beta} & \overline{\xi_\alpha\xi_\beta} & \overline{\xi_\beta^2} & \overline{\xi_\beta\xi} & \overline{\xi_\beta\eta_\alpha} & \overline{\xi_\beta\eta_\beta} \\ \overline{\eta\xi} & \overline{\xi_\alpha\xi} & \overline{\xi_\beta\xi} & \overline{\xi^2} & \overline{\xi\eta_\alpha} & \overline{\xi\eta_\beta} \\ \overline{\eta\eta_\alpha} & \overline{\xi_\alpha\eta_\alpha} & \overline{\xi_\beta\eta_\alpha} & \overline{\xi\eta_\alpha} & \overline{\eta_\alpha^2} & \overline{\eta_\alpha\eta_\beta} \\ \overline{\eta\eta_\beta} & \overline{\xi_\alpha\eta_\beta} & \overline{\xi_\beta\eta_\beta} & \overline{\xi\eta_\beta} & \overline{\eta_\alpha\eta_\beta} & \overline{\eta_\beta^2} \end{pmatrix}, \tag{5}$$

where overbars denote ensemble averages. Then P has the form

$$P(X) = \frac{\exp(-\frac{1}{2}X\Sigma^{-1}X^{\mathrm{T}})}{(2\pi)^3\sqrt{\det\Sigma}}, \tag{6}$$

where Σ^{-1} is the inverse of Σ and X^{T} the column vector corresponding to X. In writing this expression the assumption that ψ is incoherent has been employed, since all averages $\bar{X}$ have been neglected.

Two major steps are involved in evaluating the densities and fluxes $N_{\alpha\beta}$. The first is the calculation of the elements in the matrix of correlations; this will be carried out in § 4 using diffraction theory for ψ (§ 3) and averaging over the probability distribution of the random phase ϕ. The second is the evaluation of the sixfold integral (3). This will be carried out in § 5.

3. Diffraction theory

Immediately beyond $z=0$ the wave, having passed through the phase screen, is

$$\psi(\boldsymbol{R}, 0, t) = \exp[\mathrm{i}(\phi(\boldsymbol{R}, t) - \omega t)]. \tag{7}$$

For any $z>0$ the wave can be written as a sheaf of plane waves with transverse vectors $\boldsymbol{K} = (K_x, K_y)$ and frequencies ω', i.e.

$$\psi(\boldsymbol{R}, z, t) = \iint \mathrm{d}\boldsymbol{K} \int \mathrm{d}\omega' \exp\left\{\mathrm{i}\left[\boldsymbol{K}.\boldsymbol{R} + \left(\frac{\omega'^2}{c^2} - K^2\right)^{1/2} z - \omega' t\right]\right\} a(\boldsymbol{K}, \omega'). \tag{8}$$

Fourier inversion using (7) now gives $a(\boldsymbol{K}, \omega')$ as

$$a(\boldsymbol{K}, \omega') = \frac{1}{(2\pi)^3} \iint \mathrm{d}\boldsymbol{R}' \int \mathrm{d}t' \exp\{\mathrm{i}[\phi(\boldsymbol{R}', t') - \boldsymbol{K}.\boldsymbol{R}' + (\omega' - \omega)t']\}. \tag{9}$$

The approximation of quasi-monochromaticity and paraxiality consist in writing

$$\omega' \equiv \omega + \Omega \tag{10}$$

(so that Ω is a measure of departure from monochromaticity), and approximating the z wavenumber by

$$\left(\frac{(\omega+\Omega)^2}{c^2} - K^2\right)^{1/2} \approx \frac{\omega+\Omega}{c} - \frac{cK^2}{2\omega}. \tag{11}$$

This gives

$$\psi(\boldsymbol{R}, z, t)=\frac{1}{(2\pi)^3}\iint \mathrm{d}\boldsymbol{K}\int \mathrm{d}\Omega \exp[\mathrm{i}(\boldsymbol{K}.\boldsymbol{R}+\Omega z/c-cK^2z/2\omega-\Omega t)]$$
$$\times\iint \mathrm{d}\boldsymbol{R}'\int \mathrm{d}t' \exp[\mathrm{i}(\phi(\boldsymbol{R}', t')-\boldsymbol{K}.\boldsymbol{R}'+\Omega t')], \tag{12}$$

where the phase factor $\exp[\mathrm{i}(\omega z/c-\omega t)]$ has been dropped since it has no effect on the dislocations, where $\psi=0$. I shall make use of the fact that the expression (12) satisfies the 'paraxial wave equation'

$$\psi_z=-\frac{\psi_t}{c}+\frac{\mathrm{i}c}{2\omega}\nabla^2_{\boldsymbol{R}}\psi. \tag{13}$$

It will be convenient to take averages over products of ψ and ψ^* rather than ξ and η, and then obtain the elements of Σ (equation (5)) from the relations

$$\overline{\xi_\alpha\xi_\beta}=\tfrac{1}{2}\operatorname{Re}(\overline{\psi_\alpha\psi^*_\beta}+\overline{\psi_\alpha\psi_\beta}); \qquad \overline{\eta_\alpha\eta_\beta}=\tfrac{1}{2}\operatorname{Re}(\overline{\psi_\alpha\psi^*_\beta}-\overline{\psi_\alpha\psi_\beta})$$
$$\overline{\xi_\alpha\eta_\beta}=\tfrac{1}{2}\operatorname{Im}(\overline{\psi_\alpha\psi^*_\beta}+\overline{\psi_\alpha\psi_\beta}), \tag{14}$$

where the meaning of the suffices α, β has now been slightly generalised to include products like $\overline{\xi_\alpha\xi}$ where a factor is not differentiated.

4. Calculating the matrix of correlations

The final ingredient in the calculation of Σ is a specification of the statistics of the phase screen $\phi(\boldsymbol{R}, t)$. The necessary quantities are the variance $\overline{\phi^2}$ and the autocorrelation $C(\boldsymbol{\rho}, \tau)$ defined by

$$C(\boldsymbol{\rho}, \tau)\equiv\overline{\phi(\boldsymbol{R}, t)\phi(\boldsymbol{R}+\boldsymbol{\rho}, t+\tau)}/\overline{\phi^2}. \tag{15}$$

The mean products in (14) of wavefunctions (12) involve the following two averages over the Gaussian distribution of ϕ:

$$\overline{\exp[\mathrm{i}(\phi(\boldsymbol{R}_1, t_1)\mp\phi(\boldsymbol{R}_2, t_2))]}=\mathrm{e}^{-\overline{\phi^2}}\exp(\pm\overline{\phi^2}C(\boldsymbol{R}_1-\boldsymbol{R}_2, t_1-t_2)) \tag{16}$$

(the upper and lower signs refer to products $\overline{\psi_\alpha\psi^*_\beta}$ and $\overline{\psi_\alpha\psi_\beta}$ respectively).

Consider first $\overline{\psi_\alpha\psi^*_\beta}$. Formation of the product using the diffraction integral (12) leads to a twelvefold integral. Averaging with the aid of (16), together with obvious changes of variable, makes six of the integrations easy, and leads to

$$\overline{\psi_\alpha\psi^*_\beta}=\frac{\mathrm{e}^{-\overline{\phi^2}}}{(2\pi)^3}\iint \mathrm{d}\boldsymbol{K}\int \mathrm{d}\Omega[\exp[\mathrm{i}(\boldsymbol{K}.\boldsymbol{R}-\Omega t+\Omega z/c-cK^2z/2w)]]_\alpha$$
$$\times[\exp[\mathrm{i}(\boldsymbol{K}.\boldsymbol{R}-\Omega t+\Omega z/c-cK^2z/2w)]]_\beta$$
$$\times\iint \mathrm{d}\boldsymbol{\rho}\int \mathrm{d}\tau \exp[\mathrm{i}(\Omega\tau-\boldsymbol{K}.\boldsymbol{\rho}+\overline{\phi^2}C(\boldsymbol{\rho}, \tau))]. \tag{17}$$

The differentiations with respect to α and β bring down from the exponents a factor denoted by $F_{\alpha\beta}(\boldsymbol{K}, \Omega)$, and on re-arrangement this gives

$$\overline{\psi_\alpha \psi_\beta^*} = \frac{\mathrm{e}^{-\overline{\phi^2}}}{(2\pi)^3} \iint \mathrm{d}\boldsymbol{\rho} \int \mathrm{d}\tau \exp(\overline{\phi^2} C(\boldsymbol{\rho}, \tau)) \iint \mathrm{d}\boldsymbol{K} \int \mathrm{d}\Omega\, F_{\alpha\beta}(\boldsymbol{K}, \Omega) \exp[\mathrm{i}(\Omega\tau - \boldsymbol{K} \cdot \boldsymbol{\rho})]$$

$$= \mathrm{e}^{-\overline{\phi^2}} \iint \mathrm{d}\boldsymbol{\rho} \int \mathrm{d}\tau \exp(\overline{\phi^2} C(\boldsymbol{\rho}, \tau)) F_{\alpha\beta}\left(\mathrm{i}\nabla_{\boldsymbol{\rho}}, -\mathrm{i}\frac{\partial}{\partial \tau}\right) \delta(\boldsymbol{\rho}) \delta(\tau)$$

$$= \mathrm{e}^{-\overline{\phi^2}} \left[F_{\alpha\beta}\left(-\mathrm{i}\nabla_{\boldsymbol{\rho}}, \mathrm{i}\frac{\partial}{\partial \tau}\right) \exp(\overline{\phi^2} C(\boldsymbol{\rho}, \tau)) \right]_{\boldsymbol{\rho}=0, \tau=0} \tag{18}$$

where the last step results from integrating by parts.

Now each combination α, β must be treated separately. For the simplest product, $\overline{\psi\psi^*}$, it is obvious that $F = 1$, so

$$\overline{\psi\psi^*} = 1. \tag{19}$$

The product $\overline{\psi_x \psi^*}$ involves $\alpha = x$, $\beta = 0$ and (17) reveals that

$$F_{x0} = \mathrm{i}\, K_x. \tag{20}$$

(18) now gives

$$\overline{\psi_x \psi^*} = \overline{\phi^2} \frac{\partial}{\partial x} C(\mathbf{0}, 0). \tag{21}$$

But this is zero, since for the smooth phase screens considered here C, when expanded in a polynomial in x, y and τ, contains only terms of even order. Similar arguments give

$$\overline{\psi_x \psi^*} = \overline{\psi_y \psi^*} = \overline{\psi_t \psi^*} = 0. \tag{22}$$

Next, for the combination $\alpha = x_l$, $\beta = x_m$ where l or m denote x or y,

$$F_{x_l x_m} = K_l K_m, \tag{23}$$

and (18) gives

$$\overline{\psi_l \psi_m^*} = -\overline{\phi^2} \frac{\partial^2}{\partial x_l \partial x_m} C(\mathbf{0}, 0) \equiv -\overline{\phi^2}\, C_{lm} \tag{24}$$

where the obvious simplifying notation C_{lm} has been introduced. Similar arguments give

$$\overline{\psi_l \psi_t^*} = -\overline{\phi^2} C_{lt}; \qquad \overline{\psi_t \psi_t^*} = -\overline{\phi^2} C_{tt}. \tag{25}$$

The averages for which α or β is z can be evaluated with the aid of the paraxial equation (13). For $\overline{\psi_z \psi^*}$,

$$F_{z0} = \mathrm{i}\frac{\Omega}{c} - \frac{\mathrm{i}c}{2\omega} K^2, \tag{26}$$

leading to

$$\overline{\psi_z \psi^*} = +\mathrm{i}\frac{\overline{\phi^2} c}{2\omega} \nabla_{\boldsymbol{\rho}}^2 C. \tag{27}$$

For $\overline{\psi_z\psi_t^*}$,

$$F_{zt} = -\frac{\Omega^2}{c} + \frac{c\Omega K^2}{2\omega}. \tag{28}$$

The cubic term gives zero contribution, so that

$$\overline{\psi_z\psi_t^*} = \frac{\overline{\phi^2}}{c} C_{tt}. \tag{29}$$

Similarly,

$$\overline{\psi_z\psi_l^*} = \frac{\overline{\phi^2}}{c} C_{lt}. \tag{30}$$

For the final average of this type, namely $\overline{\psi_z\psi_z^*}$,

$$F_{zz} = \left(\frac{\Omega}{c} - \frac{cK^2}{2\omega}\right)^2, \tag{31}$$

and a lengthy but straightforward calculation based on (18) gives

$$\overline{\psi_z\psi_z^*} = -\frac{\overline{\phi^2}C_{tt}}{c^2} + \frac{c^2}{4\omega^2}(\overline{\phi^2})^2(3C_{xx}^2 + 3C_{yy}^2 + 4C_{yx}^2 + 2C_{xx}C_{yy}), \tag{32}$$

after a term of lower order in $\overline{\phi^2}$ has been neglected.

According to (14) the averages $\overline{\xi_\alpha\eta_\beta}$ etc involved in Σ (equation (5)) depend on averages $\overline{\psi_\alpha\psi_\beta}$ as well as on the averages $\overline{\psi_\alpha\psi_\beta^*}$ just calculated. It follows from (16), however, that these averages are of order $\exp(-2\overline{\phi^2})$ and hence negligible for the incoherent waves considered here.

When the elements of the matrix of correlations Σ are evaluated using (14) and the averages (19)–(32) it is found to be block diagonal. Thus it is natural to write

$$\Sigma = \begin{pmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{pmatrix}, \tag{33}$$

where

$$\Delta_\pm = \tfrac{1}{2}\begin{pmatrix} \overline{\psi\psi^*} & \mp \mathrm{Im}\,\overline{\psi_\alpha\psi^*} & \mp \mathrm{Im}\,\overline{\psi_\beta\psi^*} \\ \mp \mathrm{Im}\,\overline{\psi_\alpha\psi^*} & \overline{\psi_\alpha\psi_\alpha^*} & \mathrm{Re}\,\overline{\psi_\alpha\psi_\beta^*} \\ \mp \mathrm{Im}\,\overline{\psi_\beta\psi^*} & \mathrm{Re}\,\overline{\psi_\alpha\psi_\beta^*} & \overline{\psi_\beta\psi_\beta^*} \end{pmatrix}. \tag{34}$$

5. Evaluating the densities and fluxes

The fundamental quantities $N_{\alpha\beta}$ are given by equation (3) as a sixfold integral. The integrations over ξ and η can easily be performed, and use of (6), (33) and (34) gives

$$N_{\alpha\beta} = \frac{1}{(2\pi)^3 \det \Delta_+} \int \cdots \int \mathrm{d}\xi_\alpha\, \mathrm{d}\xi_\beta\, \mathrm{d}\eta_\alpha\, \mathrm{d}\eta_\beta \left| YSY^{\mathrm{T}} \right| \exp(-\tfrac{1}{2} Y M^{-1} Y^{\mathrm{T}}), \tag{35}$$

where

$$Y \equiv (\xi_\alpha, \xi_\beta, \eta_\alpha, \eta_\beta), \tag{36}$$

$$M^{-1} \equiv \begin{pmatrix} (\Delta_+^{-1})_{22} & (\Delta_+^{-1})_{23} & 0 & 0 \\ (\Delta_+^{-1})_{23} & (\Delta_+^{-1})_{33} & 0 & 0 \\ 0 & 0 & (\Delta_-^{-1})_{22} & (\Delta_-^{-1})_{23} \\ 0 & 0 & (\Delta_+^{-1})_{23} & (\Delta_-^{-1})_{33} \end{pmatrix}, \tag{37}$$

and

$$S \equiv \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{38}$$

(In writing (35) use has been made of the fact, obvious from (34), that $\det \Delta_+ = \det \Delta_- = \sqrt{\det \Sigma}$.)

To bring the integral (35) into a simple form the variables are transformed by simultaneously diagonalising the two quadratic forms YSY^{T} and $YM^{-1}Y^{\mathrm{T}}$. This gives

$$N_{\alpha\beta} = \frac{(\det M)^{1/4}}{(2\pi)^3 \det \Delta_+} \int \ldots \int \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}r\, \mathrm{d}s |\lambda_1 p^2 + \lambda_2 q^2 + \lambda_3 r^2 + \lambda_4 s^2| \exp[-\tfrac{1}{2}(p^2 + q^2 + r^2 + s^2)], \tag{39}$$

where $\lambda_1, \ldots, \lambda_4$ are eigenvalues defined by

$$\det(SM - \lambda I) = 0. \tag{40}$$

Explicit calculation then shows that

$$\lambda_1 = \lambda_2 = \tfrac{1}{2}\sqrt{\det M}; \qquad \lambda_3 = \lambda_4 = -\tfrac{1}{2}\sqrt{\det M}. \tag{41}$$

(This result depends on the fact, which follows from (34), that the '+' and '−' elements in (37) are equal.) Introduction of polar coordinates in the pq and rs planes, with radial variables

$$u \equiv \tfrac{1}{2}(p^2 + q^2), \qquad v = \tfrac{1}{2}(r^2 + s^2), \tag{42}$$

gives

$$N_{\alpha\beta} = \frac{(\det M)^{3/4}}{2\pi \det \Delta_+} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v |u - v| \,\mathrm{e}^{-(u+v)}. \tag{43}$$

Tedious elementary calculations show that the value of the double integral is unity.

The evaluation of Δ_+ and $\det M$ is simplified on realising that since α and β must be different it is always possible to choose $\alpha \neq z$. Then (22) gives

$$\mathrm{Im}\, \overline{\psi_\alpha \psi^*} = 0, \tag{44}$$

which together with (19) leads to

$$\det \Delta_+ = \tfrac{1}{8}\{\overline{\psi_\alpha \psi_\beta^*}[\overline{\psi_\beta \psi_\beta^*} - (\mathrm{Im}\, \overline{\psi_\beta \psi^*})^2] - (\mathrm{Re}\, \psi_\alpha \psi_\beta^*)^2\}. \tag{45}$$

From (37),

$$(\det M)^{3/4} = [(\Delta_+^{-1})_{22}(\Delta_+^{-1})_{33} - ((\Delta_+^{-1})_{23})^2]^{-3/2}. \tag{46}$$

The elements of Δ^{-1}, when calculated from (34) using (19) and (44), lead to

$$(\det M)^{3/4} = (2 \det \Delta_+)^{3/2}. \tag{47}$$

Substitution into (43) now gives

$$N_{\alpha\beta}=\frac{1}{2\pi}\{\overline{\psi_\alpha\psi_\alpha^*}[\overline{\psi_\beta\psi_\beta^*}-(\text{Im }\psi_\beta\psi^*)^2]-(\text{Re }\psi_\alpha\psi_\beta^*)^2\}^{1/2}. \tag{48}$$

The various densities and fluxes can now be obtained explicitly by use of equations (22)–(32), with the following results:

$$N_{xy}=\frac{\overline{\phi^2}}{2\pi}(C_{xx}C_{yy}-C_{xy}^2)^{1/2}, \tag{49}$$

$$N_{xz}=\frac{\overline{\phi^2}}{2\pi}\left(\frac{C_{xx}C_{tt}}{c^2}-\frac{C_{xt}^2}{c^2}-\frac{2\overline{\phi^2}C_{xx}c^2}{4\omega^2}(C_{xx}^2+C_{yy}^2+2C_{yx}^2)\right)^{1/2}, \tag{50}$$

$$N_{xt}=\frac{\overline{\phi^2}}{2\pi}\left(C_{xx}C_{tt}-C_{xt}^2\right)^{1/2}, \tag{51}$$

$$N_{zt}=\frac{c}{2\pi\,.\,2\omega}(\overline{\phi^2})^{3/2}[-2C_{tt}(C_{xx}^2+C_{yy}^2+2C_{yx}^2)]^{1/2} \tag{52}$$

(and similar equations for N_{yz} and N_{yt}). These formulae are the main results of this paper. The quantities in the square roots are never negative; this follows on realising that the autocorrelation function (15) decays from unity in every direction from the origin in x, y, t space, and also that combinations $C_{\alpha\alpha}C_{\beta\beta}-C_{\alpha\beta}^2$ represent the Gaussian curvature of C as a function of α and β and hence are never negative.

6. Discussion

To extract the physical content of the fundamental equations (49)–(52) it is helpful first to consider the case where the phase screen is cross-spectrally pure, so that the variations in x, y and t are independent and

$$\begin{aligned} &C_{xx}\equiv -L_x^{-2}, \qquad C_{yy}=-L_y^{-2}, \qquad C_{tt}=-T^{-2} \\ &C_{xy}=C_{xt}=C_{yt}=0. \end{aligned} \tag{53}$$

Thus L_x, L_y are the correlation lengths of the screen in the x and y directions and T is its correlation time. Next the variance $\overline{\phi^2}$ of the phase is written as

$$\overline{\phi^2}\equiv\frac{\omega^2}{c^2}S^2, \tag{54}$$

so that S is a measure of the amplitude of the undulations in wavefronts just beyond the screen. Finally, the densities and fluxes $N_{\alpha\beta}$ will be replaced by dimensionless quantities $\mathcal{N}_{\alpha\beta}$ where $\mathcal{N}_{ij}$ is the number of dislocation lines piercing a square of side equal to one wavelength, $2\pi c/\omega$, in the ij plane and $\mathcal{N}_{it}$ is the number of dislocations crossing one wavelength of the i direction in one wave period, $2\pi/\omega$. $\mathcal{N}_{\alpha\beta}$ can be written entirely in terms of the lengths L_x, L_x, cT and S as follows:

$$\mathcal{N}_{xy}=2\pi S^2/L_xL_y, \tag{55}$$

$$\mathcal{N}_{xz}=2\pi S^2[1+\tfrac{1}{2}S^2c^2T^2(L_x^{-4}+L_y^{-4})]^{1/2}/L_xcT, \tag{56}$$

$$\mathcal{N}_{xt}=2\pi S^2/L_xcT, \tag{57}$$

$$\mathcal{N}_{zt} = 2\pi S^3[\tfrac{1}{2}(L_x^{-4} + L_y^{-4})]^{1/2}/cT. \tag{58}$$

An immediate conclusion from these formulae is that all quantities $\mathcal{N}_{\alpha\beta}$ are small compared with unity. This follows from the conditions C has to satisfy in order that ψ be paraxial and quasi-monochromatic, namely

$$S/L_x \ll 1; \qquad S/L_y \ll 1; \qquad S/cT \ll 1 \tag{59}$$

(physically the wavefronts just beyond the screen have small slopes and 'shiver' much more slowly than c relative to the unperturbed wavefronts). Therefore the waves considered here are quite weakly dislocated—they appear like travelling plane waves for many wavelengths and periods around a typical event.

Another conclusion concerns the average angle θ made by the dislocations with the z axis. In the language of crystal physics, if θ is small the dislocations are predominantly of 'screw' character (Nye and Berry 1974), while if θ is near $\pi/2$ they are predominantly of 'edge' character. A measure of θ is

$$\theta \approx \cos^{-1}\left(\frac{\mathcal{N}_{xy}}{(\mathcal{N}_{xy}^2 + \mathcal{N}_{xz}^2 + \mathcal{N}_{yz}^2)^{1/2}}\right), \tag{60}$$

which by virtue of (55) and (56) is

$$\theta \approx \cos^{-1}\left[\left(1 + \frac{(L_x^2 + L_y^2)}{c^2T^2} + \frac{S^2}{2}(L_x^2 + L_y^2)(L_x^{-4} + L_y^{-4})\right)^{-1/2}\right]. \tag{61}$$

Consider first the case where the screen is "corrugated' in the x direction, i.e. $L_y \to \infty$. Then (61) gives $\theta = \pi/2$ so that all dislocations are of edge type (this is intuitively obvious). Next consider the case where the phase screen is isotropically disordered, i.e.

$$L_x = L_y = L. \tag{62}$$

Then (61) becomes

$$\theta \approx \cos^{-1}\left[\left(1 + \frac{2L^2}{c^2T^2} + \frac{2S^2}{L^2}\right)^{-1/2}\right]. \tag{63}$$

For a static screen $T \to \infty$ so that with use of (59) this becomes

$$\theta \approx \sqrt{2}\, S/L \tag{64}$$

which is small, so that for an isotropic static screen the dislocations are predominantly of screw type. However, if the isotropic screen is not static, but moves so fast that $cT \ll L$ (a condition not violating (59)), then

$$\theta \approx \frac{\pi}{2} - \frac{cT}{L\sqrt{2}}, \tag{65}$$

so that the dislocations are now predominantly of edge type.

Finally, it is interesting to show the consistency of the basic formulae (49)–(52) when the phase screen is moving rigidly along the x axis with speed v. It is then not cross-spectrally pure, and

$$C(x, y, t) = C(x - vt, y). \tag{66}$$

In this case the pattern of dislocation lines simply translates rigidly in the x direction. Therefore N_{xt} (equation (51)) must vanish, and indeed this follows from (66), which implies that $C_{xx}C_{tt} - C_{xt}^2$ is zero. Also N_{xy} and N_{xz} must be independent of v, and this also follows from (66). Lastly, the densities and fluxes must obey the 'continuity equations'

$$N_{zt} = vN_{xz}; \qquad N_{yt} = vN_{xy}, \tag{67}$$

and this also follows from (66).

7. Concluding remarks

It would be desirable to extend the analysis of this paper in several directions.

In the first place, the transmission of quasi-monochromatic *pulses* with static screens should be studied. For incoherent waves the result (48) will still hold if the term $(\mathrm{Im}\, \psi_\beta \psi^*)^2$ and also the whole expression in curly brackets are divided by $\overline{\psi\psi^*}$ (to make all three terms dimensionless with respect to ψ). But the averages $\overline{\psi_\alpha \psi_\beta^*}$ etc will no longer be given by the expressions in § 4; they could be evaluated, for example, by generalising techniques I introduced in an earlier study of $\overline{\psi\psi^*}$ for pulses reflected by rough surfaces (Berry 1973). This generalisation to pulses is required to study dislocations produced during echo sounding.

In the second place, the restriction of incoherence (i.e. $|\bar{\psi}|^2 \ll \overline{|\psi|^2}$) should be relaxed, to enable a description to be given of the 'healing' of the disrupted wavefronts as $\overline{\phi^2} \to 0$. Preliminary arguments suggest that the densities and fluxes will contain a factor

$$\exp\left(-\frac{\exp(-\overline{\phi^2})}{[1-\exp(-\overline{\phi^2})]^2}\right) \to \begin{cases} 1-\mathrm{e}^{-\overline{\phi^2}} & \text{as } \overline{\phi^2} \to \infty \text{ (incoherent)} \\ \exp[-(\overline{\phi^2})^{-2}] & \text{as } \overline{\phi^2} \to 0 \text{ (coherent)} \end{cases} \tag{68}$$

suggesting that the dislocations are eliminated very rapidly as $\overline{\phi^2}$ gets small and the wave becomes coherent.

In the third place, the restriction to Gaussian randomness should be relaxed. This would enable a description to be given of the dislocations for z in the focusing regime, which is important when $\overline{\phi^2}$ is large since it is a transition zone, where the wave is non-Gaussian (Berry 1977), between incoherent ($z \to \infty$) and coherent ($z \to 0$) Gaussian behaviour. I conjecture that for a static phase screen with near-isotropic disorder most of the dislocations come into the focusing regime from large z (where they have predominantly screw character, as shown in § 6), turn over like 'hairpins' near elliptic and hyperbolic umbilic catastrophe focal points (Berry 1976, Berry and Hannay 1977) and then retreat back to large z. This conjecture is based on the results of a detailed experimental, analytical and computational study of the elliptic umbilic diffraction catastrophe made in conjunction with J F Nye and F J Wright (unpublished).

In the fourth place, the statistical topology of the dislocations should be investigated. Do most of them form closed loops or are they of infinite length? When the screen is time dependent do dislocations encountering one another simply pass through without interaction, or do they change topology?

And finally, the restrictions of paraxiality and quasi-monochromaticity should be relaxed. One interesting problem that would then become amenable to study is the dislocation structure of black-body radiation.

Acknowledgments

I thank Professor J F Nye for many helpful discussions on this subject, and Dr J H Hannay for the useful suggestion that the calculations would be simplified by employing the paraxial wave equation (13).

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